CHAPTER

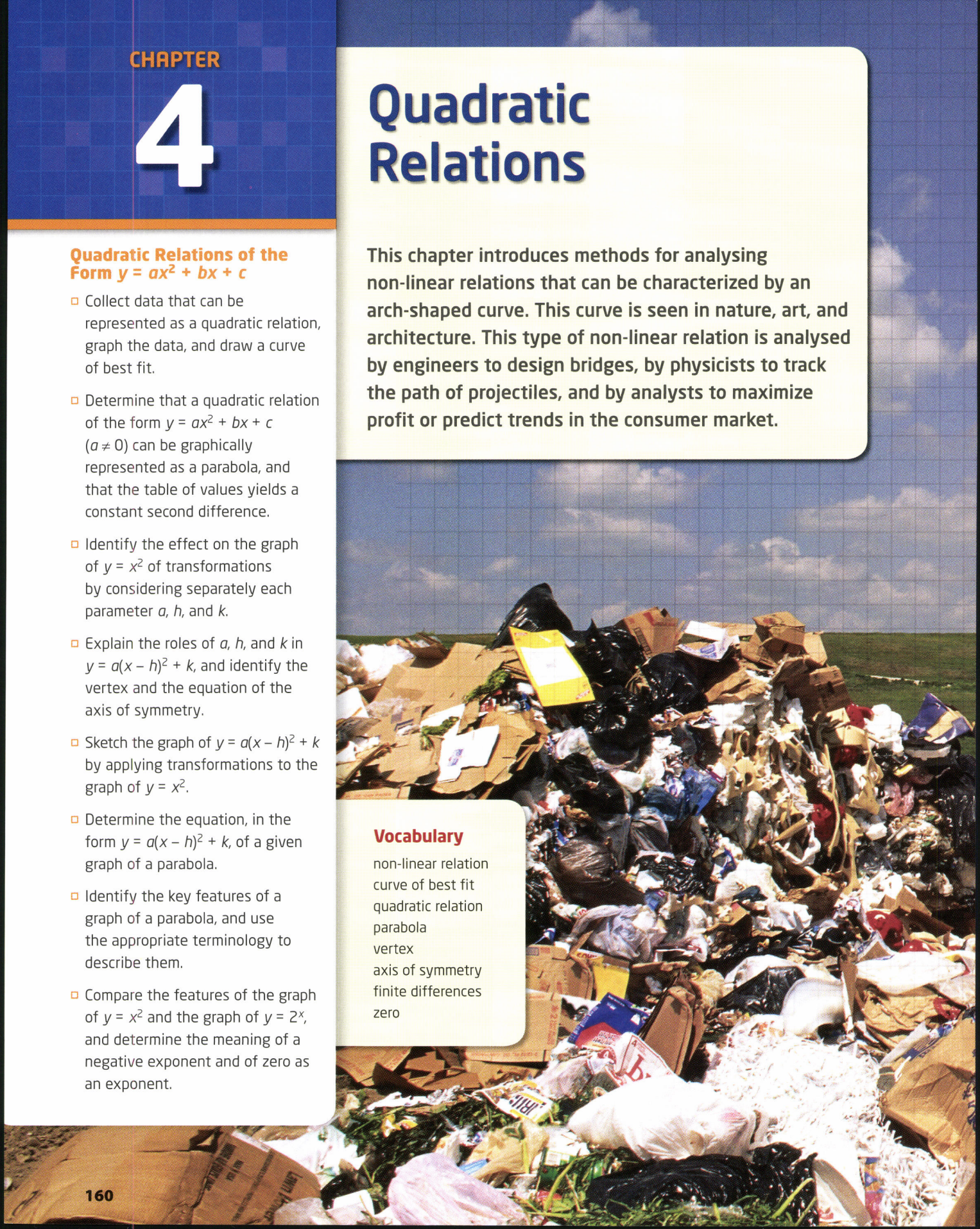
4

Quadratic Relations

This chapter introduces methods for analysing non-linear relations that can be characterized by an arch-shaped curve. This curve is seen in nature, art, and architecture. This type of non-linear relation is analysed by engineers to design bridges, by physicists to track the path of projectiles, and by analysts to maximize profit or predict trends in the consumer market.

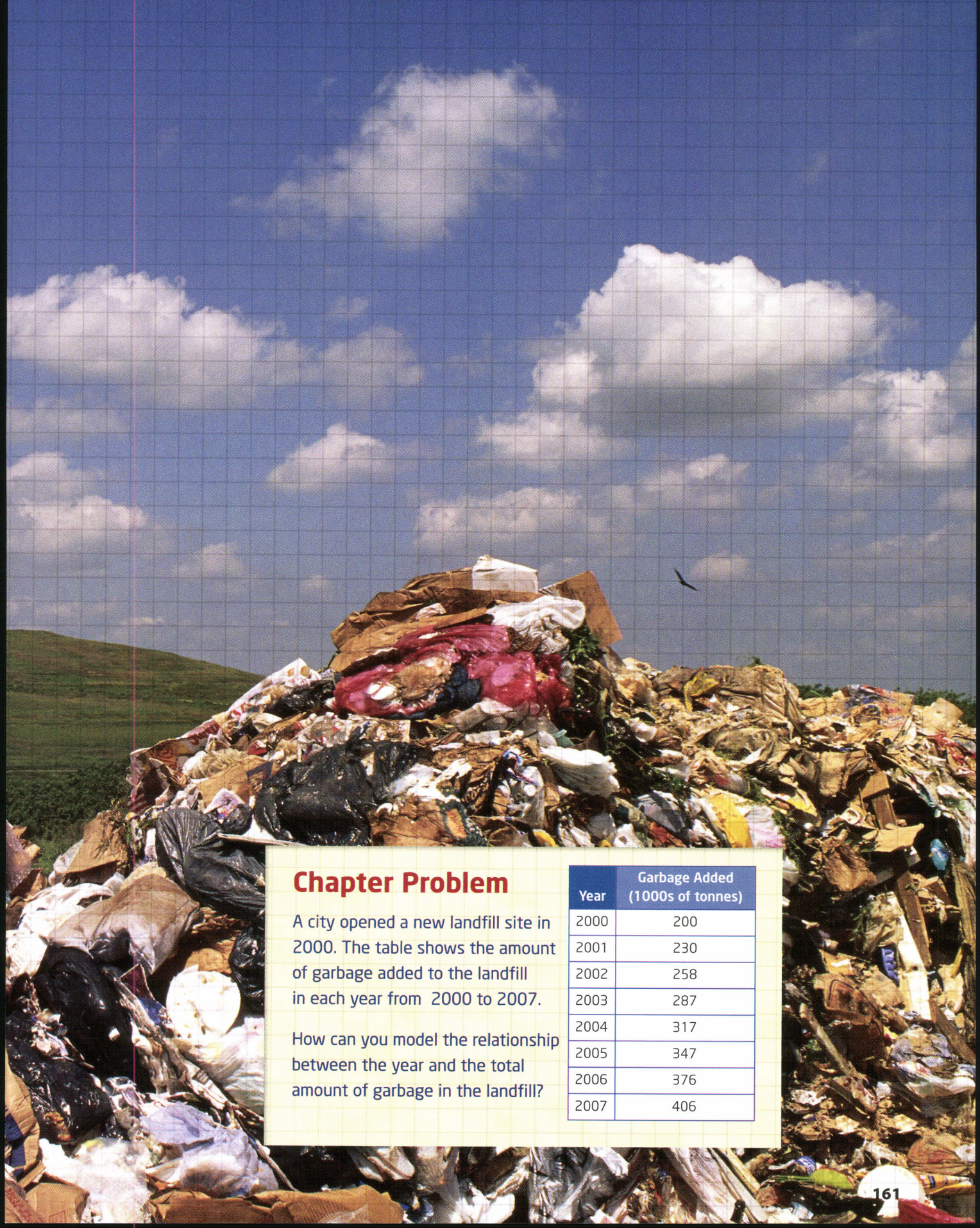
Quadratic Relations of the Form $y = ax^2 + bx + c$

- Collect data that can be represented as a quadratic relation, graph the data, and draw a curve of best fit.
- Determine that a quadratic relation of the form $y = ax^2 + bx + c$ ($a \neq 0$) can be graphically represented as a parabola, and that the table of values yields a constant second difference.
- Identify the effect on the graph of $y = x^2$ of transformations by considering separately each parameter a, h, and k.
- Explain the roles of a, h, and k in $y = a(x - h)^2 + k$, and identify the vertex and the equation of the axis of symmetry.
- Sketch the graph of $y = a(x - h)^2 + k$ by applying transformations to the graph of $y = x^2$.
- Determine the equation, in the form $y = a(x - h)^2 + k$, of a given graph of a parabola.
- Identify the key features of a graph of a parabola, and use the appropriate terminology to describe them.
- Compare the features of the graph of $y = x^2$ and the graph of $y = 2^x$, and determine the meaning of a negative exponent and of zero as an exponent.



Vocabulary

non-linear relation
curve of best fit
quadratic relation
parabola
vertex
axis of symmetry
finite differences
zero



Chapter Problem

A city opened a new landfill site in 2000. The table shows the amount of garbage added to the landfill in each year from 2000 to 2007.

How can you model the relationship between the year and the total amount of garbage in the landfill?

Year	Garbage Added (1000s of tonnes)
2000	200
2001	230
2002	258
2003	287
2004	317
2005	347
2006	376
2007	406

Get Ready

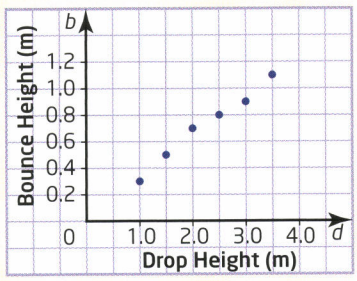
Scatter Plots

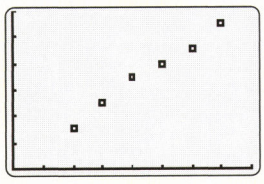
An experiment measured how high a ball bounced after being dropped from six different heights.

Drop Height (m)	1.0	1.5	2.0	2.5	3.0	3.5
Bounce Height (m)	0.3	0.5	0.7	0.8	0.9	1.1

The independent variable is the drop height because it is being controlled in the experiment. So, the bounce height is the dependent variable.

The scatter plot shows that the relationship between the two variables is approximately linear.





If the trend in the scatter plot continues, the bounce height of a ball that is dropped from 4.0 m will be approximately 1.2 m.

Making Connections

Refer to the Technology Appendix for help with creating a scatter plot on a TI-83 Plus or TI-84 Plus graphing calculator.

1. The table shows the height of a tomato plant during the first 2 weeks after it germinated.

Time (days)	1	2	3	4	5	6	7
Height (cm)	0.4	1.2	2.5	3.4	4.3	5.2	6.5
Time (days)	8	9	10	11	12	13	14
Height (cm)	7.5	8.5	9.3	10.3	11.2	12.4	13.4

a) Identify the independent variable and the dependent variable.

b) Make a scatter plot of the data.

c) Describe the relationship between the variables.

d) Predict the height of the plant after 17 days.

2. The table shows the heights and neck circumferences of 12 students.

Height (cm)	157	168	162	151	157	170
Neck Circumference (cm)	32	40	36	31	34	42
Height (cm)	167	159	168	171	176	154
Neck Circumference (cm)	38	38	38	40	40	32

a) Identify the independent variable and the dependent variable.

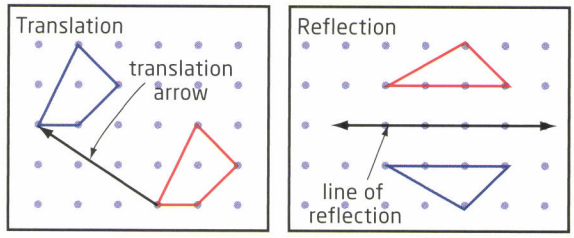
b) Make a scatter plot of the data.

c) Describe the relationship between the variables.

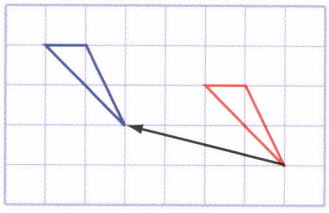
d) Predict the neck circumference for a 180-cm-tall student.

Translations and Reflections

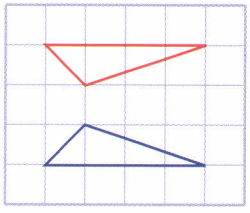
Translations and reflections are two types of transformations that move one geometric figure onto another.



3. Describe the translation of the red figure to the blue figure.



4. Copy the diagram. Show the location of the line of reflection.



Operations With Powers

A power is a product of identical factors and consists of a base and an exponent.

2^4 is a power.

base exponent

The exponent laws can be used to simplify expressions involving powers with the same base.

Operation	Exponent Rule	Example
multiplication	$x^a \times x^b = x^{a+b}$	$3^2 \times 3^3 = 3^{2+3} = 3^5$
division	$x^a \div x^b = x^{a-b}$	$3^5 \div 3^2 = 3^{5-2} = 3^3$
powers	$(x^a)^b = x^{a \times b}$	$(3^2)^4 = 3^{2 \times 4} = 3^8$

5. Use the exponent laws to write each as a single power.

a) $2^3 \times 2^4$

b) $(-1)^2 \times (-1)^5$

c) $\left(\frac{1}{2}\right)^2 \times \left(\frac{1}{2}\right)^3$

d) $5^8 \div 5^3$

e) $(-3)^7 \div (-3)^4$

f) $(4^2)^5$

6. Use the exponent laws to write each as a single power.

a) $2^3 \times 2^4 \div 2^5$

b) $(-3)^9 \div (-3)^5 \times (-3)^2$

c) $(5^2)^4 \div 5^3$

d) $4^7 \times 4^3 \div (4^2)^4$

4.1 Investigate Non-Linear Relations

non-linear relation

- a relationship between two variables that does not follow a straight line when graphed

Most relations that you have studied in mathematics have been linear. However, many **non-linear relations** also exist in real life. For example, the area of a shape is measured in square units, so the graph of area versus length is non-linear. Similarly, when a rocket is launched, it follows an arch-like path.

Investigate

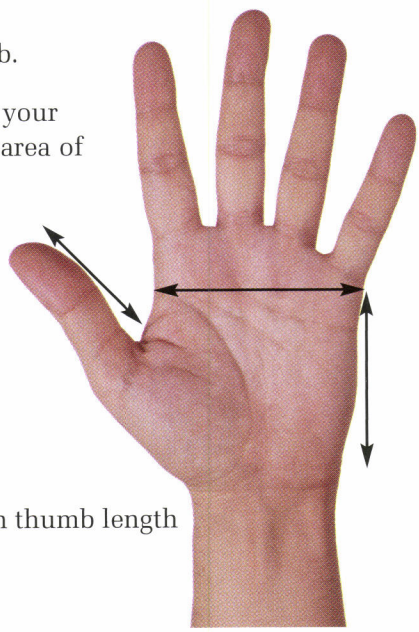
Tools

- ruler
- grid paper

How can you use a scatter plot to model non-linear data?

A: Relate Thumb Length and Palm Area

Work in small groups.



1. Measure the length of your thumb.
2. Measure the length and width of your palm. Calculate the approximate area of your palm.
3. Record the thumb length and palm area data for each group member.
4. Identify the independent and dependent variables.
5. Make a scatter plot of the data.
6. Describe the relationship between thumb length and palm area.
7. Draw a **curve of best fit**.
8. **Reflect** Why is a curve of best fit used for these data instead of a line of best fit?
9. Use your model to predict the area of a person's palm when that person's thumb is 8.1 cm long.

curve of best fit

- a smooth curve drawn to approximate the general path or trend in a scatter plot

B: Relate Distance and Roll Time

Work in small groups.

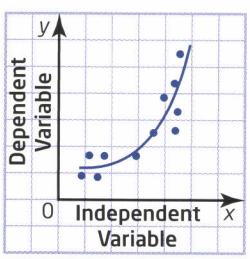
Tools

- board at least 1.2 m in length
- 2 textbooks
- 1 can of soup (or other object that rolls)
- metre stick or measuring tape
- stopwatch
- grid paper

1. Build a ramp using two textbooks as a support.
2. Place a can 30 cm from the bottom of the ramp.
3. Release the can and time how long it takes the can to roll to the bottom. You may wish to practise starting the stopwatch at the exact moment the can is released.
4. Repeat by releasing the can from 40 cm, 50 cm, and so on.
5. Record all your data in a table. Choose your variables.
6. Identify the independent and dependent variables. Explain your choices.
7. Make a scatter plot of the results.
8. Describe the relationship between distance and roll time.
9. Draw a curve of best fit.
10. **Reflect** Why is a curve of best fit used for these data instead of a line of best fit?
11. Use your model to predict the time it would take for a can to roll down a longer ramp with the same slope from a distance of 160 cm.

Key Concepts

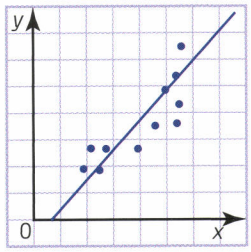
- The independent variable is the one that you control before the trial begins. The dependent variable is the one that you measure during the trial. It is affected by a change in the independent variable.
- Look at the pattern of the points in a scatter plot when deciding if the relation is linear or non-linear. The points in a non-linear relation will not lie along a line, but will form a graph that is curved.



Communicate Your Understanding

C1 State whether each line or curve of best fit is a good model for the data. Justify your answer.

a)



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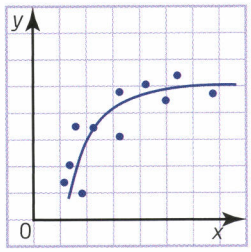
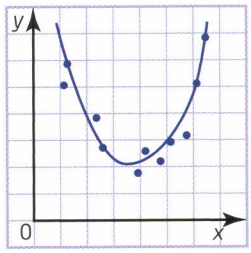
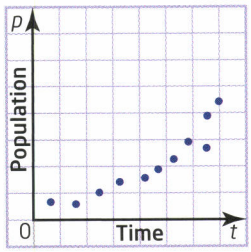


c)



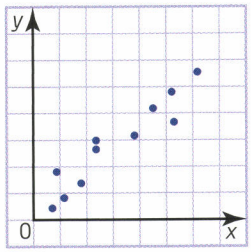
C2 The scatter plot shows the relationship between time, in 5-year intervals, and the population of a town. Explain why time was used as the independent variable.



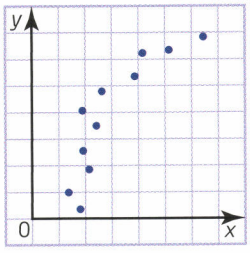
Practise

1. Which scatter plot(s) could be modelled using a curve instead of a line of best fit? Explain.

a)

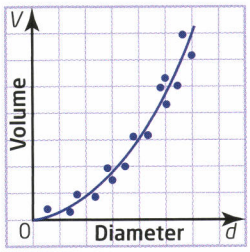


b)



2. The scatter plot and curve of best fit show the relationship between the diameter of rain-collection barrels and the volume of water collected.

Is this relation linear or non-linear? Justify your answer.



3. An altimeter is attached to a model rocket before it is launched. The table shows the recorded data from the rocket's flight.

Time (s)	1	2	3	4	5	6	7
Height (m)	230	310	350	360	350	300	220

a) Make a scatter plot of the data.

b) Describe the relation.

c) Draw a curve of best fit.

d) Use your model to predict the height of the rocket after 8 s.

4. The table shows the average fuel economy of a car at a test track.

Speed (km/h)	Fuel Economy (L/100 km)
10	14.26
20	12.85
40	10.65
60	10.10
70	10.24
80	10.84
100	12.14
120	15.64
130	16.88
150	22.50

a) Make a scatter plot of the data.

b) Describe the relation.

c) Draw a curve of best fit.

d) Use your model to predict the fuel economy at 200 km/h.

e) This car does not get very good fuel economy. How would a graph of a car with better fuel economy look? Why?

Did You Know?

New vehicles have an EnerGuide label that shows the city and highway fuel-consumption ratings and an estimated annual fuel cost for that vehicle. The fuel-consumption ratings are provided by vehicle manufacturers and are based on standardized testing procedures and driving cycles performed under controlled conditions.

5. The table shows the data for a bouncing ball.

Bounce Number	1	2	3	4	5	6	7
Rebound Height (cm)	270	180	120	80	53	45	25

a) Make a scatter plot of the data.

b) Describe the relation.

c) Draw a curve of best fit.

d) How would the relationship change for a ball that was bouncier?

6. Chapter Problem A city opened a new landfill site in 2000. The table shows how much garbage was added to the landfill in each year from 2000 to 2007.

Year	Garbage Added (1000s of tonnes)
2000	200
2001	230
2002	258
2003	287
2004	317
2005	347
2006	376
2007	406

a) Determine the total mass of garbage in the landfill at the end of each year.

b) Make a scatter plot of the total mass of garbage versus the year. Draw a curve of best fit.

c) What problems do you predict if growth continues at its current rate?

7. A rectangle has a width of x centimetres, and its length is double its width.

a) Create a table comparing the length and area of a rectangle for widths up to 8 cm.

b) Make a scatter plot of the data.

c) Draw a curve of best fit.

d) Explain why the graph of this relation is non-linear.

4.2 Quadratic Relations

The Galleria, in BCE Place in Toronto, has many arches. These curved structures are used to span a space while supporting weight. Just as a linear relation can be modelled with a linear equation, some non-linear relations, such as the shape of an arch, can be modelled using non-linear equations.

Tools

- grid paper

Investigate A

How can you compare relations of the form $y = ax^2 + bx + c$?

1. Make a table of values for each relation, using integer values of x from -3 to $+3$.
   a) $y = x^2$ b) $y = 2x^2$ c) $y = x^2 + 2x + 3$
   d) $y = -x^2$ e) $y = -0.5x^2 + 3$
2. Graph all the relations in step 1 on the same set of axes. Plot each set of ordered pairs and draw a smooth curve through the points.
3. **Reflect** Describe the graphs you created in as many ways as you can. What is similar about the graphs? What is different?

quadratic relation

- a relation whose equation is in the form $y = ax^2 + bx + c$, where a, b, and c are real numbers and $a \neq 0$

parabola

- the graph of a quadratic relation, which is U-shaped and symmetrical

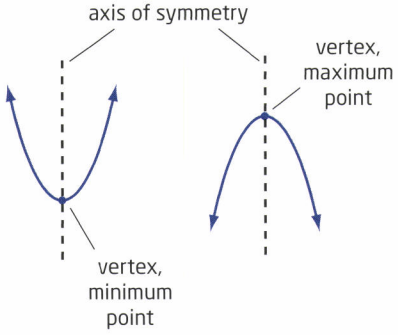
vertex

- the point on a parabola where the curve changes direction
- the maximum point if the parabola opens down
- the minimum point if the parabola opens up

axis of symmetry

- the line that divides a figure into two congruent parts

The relation described by $y = ax^2 + bx + c$ is called a **quadratic relation**. The graph of a quadratic relation is called a **parabola**. A parabola has a minimum point or a maximum point called the **vertex**. It is also symmetrical about a vertical line drawn through the vertex, called the **axis of symmetry**.



Investigate B

How can you use finite differences to determine if a relation is linear or quadratic?

finite differences

- differences found from the y-values in tables with evenly spaced x-values
- first differences are the differences between consecutive y-values, second differences are the differences between consecutive first differences, and so on

1. Copy and complete the table for each linear relation. Calculate the y-values. Then, calculate the first differences by subtracting consecutive y-values.

a) $y = 2x - 5$

x	y	First Differences
−2	−9	−7 − (−9) = 2
−1	−7	
0		
1		
2		

b) $y = -6x + 2$

x	y	First Differences
−2	14	8 − 14 = −6
−1	8	
0		
1		
2		

2. What is true about the first differences for a linear relation?

3. Copy and complete the table for each quadratic relation. Calculate the y-values and first differences. Then, calculate the second differences by subtracting successive first differences.

a) $y = x^2 - 4$

x	y	First Differences	Second Differences
−2	0		
−1	−3	−3 − 0 = −3	−1 − (−3) = 2
0	−4	−4 − (−3) = −1	
1			
2			

b) $y = 2x^2 + 3x - 1$

x	y	First Differences	Second Differences
−2	1		
−1	−2		
0			
1			
2			

4. a) What is true about the first differences for a quadratic relation?

b) What is true about the second differences for a quadratic relation?

5. Reflect Write a rule for using finite differences to determine whether a relation is linear or quadratic.

Example Galleria Arches

Each arch in the BCE Place Galleria can be approximated by the relation $y = -0.55x^2 + 26$, where y is the height, in metres, above the floor and x is the width, in metres, from the centre of the hallway.

a) Graph the quadratic relation.

b) Describe the shape of the arch.

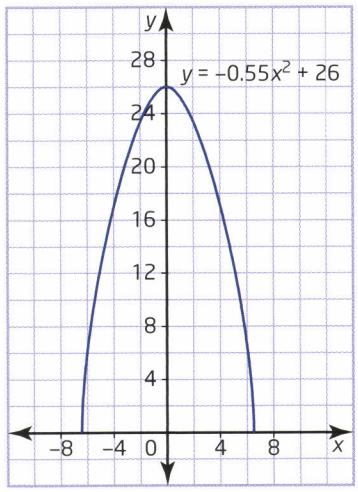
c) How tall and wide is the arch?

Solution

a) Method 1: Use Pencil and Paper

Use a table of values to help you sketch the graph.

x	y
−6	6.2
−4	17.2
−2	23.8
0	26.0
2	23.8
4	17.2
6	6.2

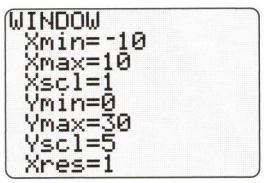


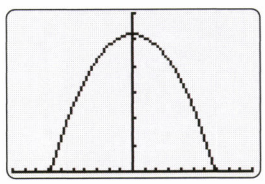
Method 2: Use a Graphing Calculator

Enter the equation using [Y=].

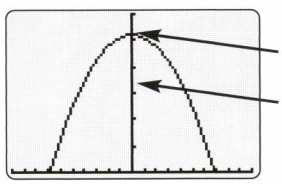
Press [WINDOW] and enter the settings shown.

Then, press [GRAPH].

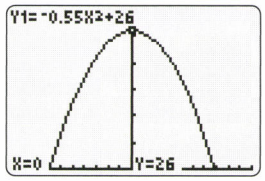




b) You can see that the shape of the arch is parabolic. The parabola is symmetrical about a vertical line, the y-axis. The graph has a maximum point.



c) You can read the maximum y-value from the pencil-and-paper graph or use the TRACE feature on the graphing calculator.



I can see that the maximum occurs when $x = 0$. From the equation, when $x = 0$, $y = 26$.

Since the maximum value of y is 26, the height of each arch is 26 m.

The x-axis represents the floor of the hallway. The width of each arch is the difference between the two x-intercepts. From the pencil-and-paper graph, the x-intercepts appear to be about 7 and -7. Use the TRACE feature on the graphing calculator to find that the curve crosses the x-axis at about -6.9 and $+6.9$.

$6.9 - (-6.9) = 13.8$

The width of each arch is about 13.8 m.

Technology Tip

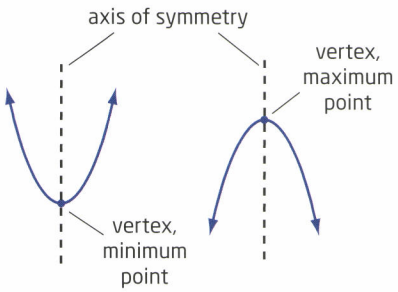
You can get a better approximation of the x-intercepts by zooming in.

- Position the cursor near one of the x-intercepts.
- Press ZOOM, select **2:Zoom In**, and then press ENTER.
- Press TRACE and reposition the cursor.

The accuracy of the approximation improves each time you repeat these steps.

Key Concepts

- The relation defined by $y = ax^2 + bx + c$ is a quadratic relation.
- The graph of a quadratic relation is called a parabola.
- The vertex of a parabola is either the minimum point or the maximum point on the graph.



- A parabola is symmetric about a vertical line that passes through the vertex. This line is the axis of symmetry.
- If a relation is quadratic, the second differences are constant, but the first differences are not.

Communicate Your Understanding

C1 El-Noor used the following incorrect technique to determine that the relation is not quadratic. Explain the flaw in his reasoning.

x	y	First Differences	Second Differences
−3	13		
		−10	
−2	3		2
		−8	
0	−5		10
		2	
1	−3		4
		6	
2	3		18
		24	
4	27		

C2 In Section 4.1, Investigate Part A, you found that the relationship between thumb length and palm area is non-linear. Is the relation quadratic? Explain.

Practise

For help with questions 1 and 2, see the Example.

1. The table shows the path of a ball, where x is the horizontal distance, in metres, and h is the height, in metres, above the ground.

x	h
0	1
1	8
2	13
3	16
4	17
5	16
6	13
7	8
8	1

a) Sketch a graph of the quadratic relation.

b) Describe the flight path of the ball. Identify the axis of symmetry and the vertex.

c) What is the maximum height that the ball reached?

d) Verify that $h = -x^2 + 8x + 1$ can be used to model the flight path of the ball.

2. The underside of a bridge is an arch that can be approximated by the relation $y = -0.1x^2 + 10$, where y is the height, in metres, above the ground and x is the width, in metres, from the centre of the bridge.

a) Graph the quadratic relation.

b) Describe the shape of the arch.

c) How tall and wide is the arch?

3. Use finite differences to determine whether each relation is linear, quadratic, or neither.

a)

x	y
0	4
1	5
2	6
3	7
4	8

b)

x	y
0	3
1	4
2	7
3	12
4	19

c)

x	y
1	0
3	1
5	8
7	27
9	64

d)

x	y
−2	6
1	0
4	12
7	42
10	90

Connect and Apply

4. This section has photographs of parabolic arches in architecture, furniture, bridge design, and nature. Find five more examples of parabolic arches. Some possible sources are the Internet, personal surroundings, or print-based material. Explain how you determined that your examples are parabolic.

5. The parabolic shape of the Humber River Pedestrian Bridge in Toronto can be approximated by the equation $h = -\frac{1}{144}x^2 + \frac{5}{6}x$, where x is the horizontal distance, in metres, from one end and h is the height, in metres, above the water.

a) Graph the quadratic relation with or without technology.

b) What is the height of the bridge 12 m horizontally from one end?

c) How wide is the bridge at its base?

d) What is the maximum height of the bridge? At what horizontal distance does it reach that height?

e) Identify the axis of symmetry of the bridge.

6. Use Technology A ball is thrown upward at an initial velocity of 15 m/s, from a height of 1.5 m. The height, h, in metres, of the ball above the ground after t seconds can be found using the relation $h = -4.9t^2 + 15t + 1.5$.

a) Graph this relation using a graphing calculator.

b) Describe the relationship between time and height.

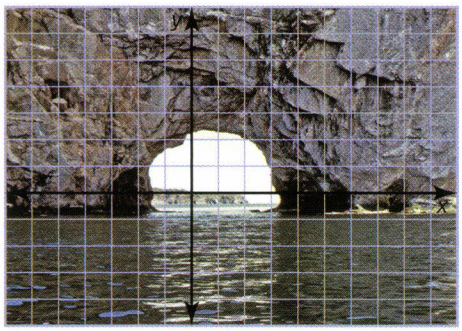
c) Repeat parts a) and b) for a ball thrown upward on the Moon, with height defined by the relation $h = -0.81t^2 + 15t + 1.5$.

d) Repeat parts a) and b) for a ball thrown upward on Jupiter, with height defined by the relation $h = -11.55t^2 + 15t + 1.5$.

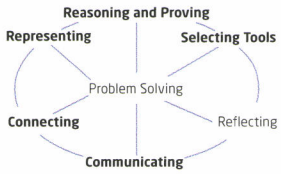
e) Compare the results from the three locations.

7. Chapter Problem A city opened a new landfill site in 2000. In Section 4.1, question 6, you created a table of values showing the total mass of garbage in the landfill for each year from 2000 to 2007. Use your table and finite differences to determine if the relationship more closely models a linear or a quadratic relation. Justify your decision.

8. Percé Rock is located at the eastern tip of Quebec's Gaspé Peninsula. Make a table of values of at least seven points, so that the x-values are equally spaced. Use finite differences to determine how close the arch is to a parabola.



Achievement Check



9. The path of a rocket fired at a Canada Day fireworks display is given by $h = -4.9t^2 + 19.6t + 0.4$, where h is the height, in metres, of the rocket above the ground and t is the time, in seconds.

a) Make a table of values for $t = 0$ to $t = 4$.

b) Make a table of first and second differences. What conclusion can you make?

c) Draw a graph of the path of the rocket using the table of values from part a) or graphing technology. Describe the path of the rocket.

d) How high above the ground was the rocket when it was set off? Explain your answer.

Extend

10. The flow rate of water through a garden hose depends on the water pressure and the diameter of the hose opening. At a normal water pressure of 345 kPa, the flow rate can be calculated using the formula $r = 2d^2$, where d is the diameter, in centimetres, of the hose opening and r is the flow rate, in litres per second. How long would it take to fill a 200-L barrel using a hose with a 0.3-cm-diameter opening?

11. The sum of the first n natural numbers is a quadratic relation. Determine that relation and verify it for the first six natural numbers.

4.3 Investigate Transformations of Quadratics

When police investigate car accidents, they often measure the length of a tire skid mark to determine the speed of the car before braking. Such calculations involve quadratic relations. On dry pavement, the length of a skid mark, l, is related to the speed of the car, s, before braking by the relation $l = 0.04s^2$.

Investigate

How do transformations of the graph of $y = x^2$ affect the equation?

Tools

- TI-83 Plus or TI-84 Plus graphing calculator
- grid paper

A: Compare the Graphs of $y = x^2$ and $y = x^2 + k$

1. First, clear any graphed equations.
   - Press [Y=] and use the [CLEAR] key to remove any equations.
   - Make sure **Plot1**, **Plot2**, and **Plot3** are not highlighted. If they are, use the [▶], [◀], [▲], and [▼] keys to move to each and press [ENTER].

2. Use a standard window.
   - Press [ZOOM] and select **6:ZStandard.**
   - You can view the window settings by pressing [WINDOW].

```
WINDOW
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
 Xres=1
```

3. Graph the equation $y = x^2$ as **Y1**.
   - Press [Y=]. Beside **Y1=**, press [X,T,θ,n] [x^2].
   - Press [GRAPH] to view the parabola.

4. Enter $y = x^2 + 2$ as **Y2** and $y = x^2 - 4$ as **Y3**. Press [GRAPH].

5. **a)** Sketch all three graphs on the same set of axes. Label each parabola with its equation.
   b) Describe the transformations.
   c) Without using a graphing calculator, sketch the graph of $y = x^2 - 8$.

6. **Reflect** Describe how the value of k in $y = x^2 + k$ changes the graph of $y = x^2$.

B: Compare the Graphs of $y = x^2$ and $y = ax^2$

1. **a)** Clear the equations from **Y2=** and **Y3=**, but keep $y = x^2$ in **Y1**.
   b) Graph the equations $y = 2x^2$ and $y = 3x^2$.

2. **a)** Sketch all three graphs on the same set of axes. Label each parabola with its equation.
   b) Describe the transformations.

3. Without using a graphing calculator, sketch the graph of $y = 4x^2$.

4. Repeat steps 1 and 2 for the equations $y = \frac{1}{2}x^2$ and $y = \frac{1}{4}x^2$.

5. Without using a graphing calculator, sketch the graph of $y = \frac{1}{3}x^2$.

6. Repeat steps 1 and 2 for the equations $y = -2x^2$ and $y = -0.5x^2$.

7. Without using a graphing calculator, sketch the graph of $y = -3x^2$.

8. **Reflect** Describe how the value of a in $y = ax^2$ changes the graph of $y = x^2$.

C: Compare the Graphs of $y = x^2$ and $y = (x - h)^2$

1. **a)** Clear all equations except $y = x^2$.
   b) Graph the equations $y = (x - 2)^2$ and $y = (x - 5)^2$.

2. **a)** Sketch all three graphs on the same set of axes. Label each parabola with its equation.
   b) Describe the transformations.

3. Without using a graphing calculator, sketch the graph of $y = (x - 3)^2$.

4. Repeat steps 1 and 2 using the equations $y = (x + 2)^2$ and $y = (x + 5)^2$.

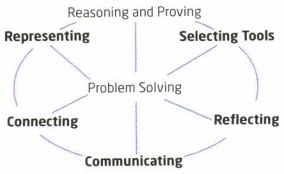
5. Without using a graphing calculator, sketch the graph of $y = (x + 3)^2$.

6. **Reflect** Describe how the value of h in $y = (x - h)^2$ changes the graph of $y = x^2$.

Technology Tip

A table of values can help you sketch the graph of an equation entered using [Y=]. You can specify how a table of values is set up.

- Press [2nd] [TBLSET] to display the **TABLE SETUP** screen. Make sure both **Indpnt** and **Depend** are set to **Auto**. Enter the desired starting x-value (**TblStart**) and x increment (**ΔTbl**). For example, try **TblStart=−10** and **ΔTbl=1**.
- Press [2nd] [TABLE] to view the table of values.



Example Falling Stone

A stone is dropped from the top of a 50-m cliff above a river. Its height, y, in metres, above the water can be estimated using the relation $y = -4.9x^2 + 50$, where x is the time, in seconds.

a) Graph the relation.

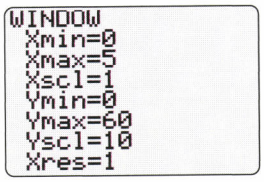
b) Find the intercepts. What do they represent?

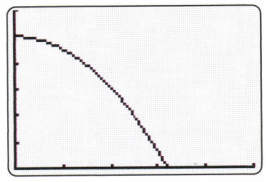
c) How would the equation change if the stone were dropped from a 75-m cliff instead of a 50-m cliff?

d) For what values of x is each equation valid?

Solution

a) Use a graphing calculator with the window settings shown.





Since height and time cannot be negative, this graph shows only part of a parabola.

b) For the y-intercept, let $x = 0$.

$y = -4.9(0)^2 + 50$

$= 50$

I can see that the y-intercept is 50 from the graph.

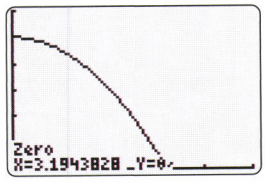
The y-intercept is 50. This represents the height from which the stone was dropped, 50 m above the water.

zero

- a value of x for which a relation has a value of 0
- corresponds to an x-intercept of the graph of the relation

To find the x-intercept, or **zero**, of the relation use the Zero operation on a graphing calculator.

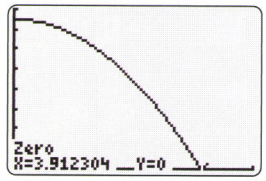
- Press [2nd] [CALC] to display the **CALCULATE** menu, and select **2:zero**.
- Move the cursor to the left of the x-intercept and press [ENTER].
- Move the cursor to the right of the x-intercept and press [ENTER].
- Press [ENTER] again.



The x-intercept is approximately 3.19.
This represents the time when the stone hits the water, 3.19 s.

c) The constant term would be 75 instead of 50. The equation would change to $y = -4.9x^2 + 75$.

d) The original equation $y = -4.9x^2 + 50$ is valid for $0 \leq x \leq 3.19$ (approximately).

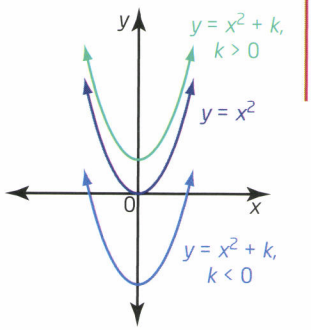


Graph the new equation and use the Zero operation. You will need to change the WINDOW settings to be able to see the whole graph.

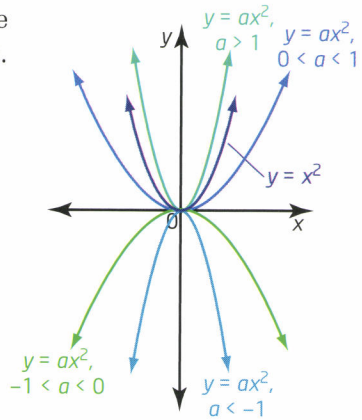
The new equation $y = -4.9x^2 + 75$ is valid for $0 \leq x \leq 3.91$ (approximately).

Key Concepts

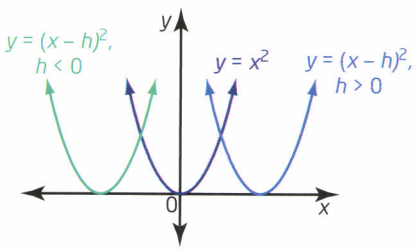
- To graph $y = x^2 + k$, translate the graph of $y = x^2$ vertically k units.
  - If $k > 0$, then the graph is translated k units upward.
  - If $k < 0$, then the graph is translated k units downward.



- To graph $y = ax^2$, stretch or compress the graph of $y = x^2$ vertically by a factor of a.
  - If $a < 0$, the parabola is reflected in the x-axis.
  - If $a > 1$ or $a < -1$, then the graph is stretched vertically (narrows).
  - If $-1 < a < 0$ or $0 < a < 1$, then the graph is compressed vertically (widens).



- To graph $y = (x - h)^2$, translate the graph of $y = x^2$ horizontally h units.
  - If $h > 0$, then the graph is translated h units to the right.
  - If $h < 0$, then the graph is translated h units to the left.



Communicate Your Understanding

C1 How do the graphs of $y = 2x^2$ and $y = -2x^2$ compare? Explain the similarities and the differences.

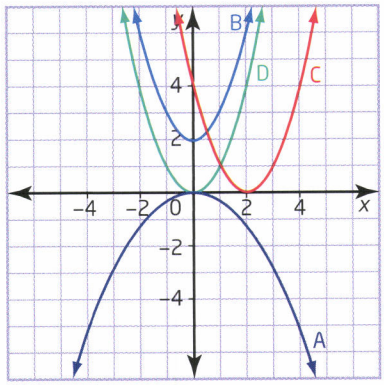
C2 Match each graph with the appropriate equation. Explain your reasoning.

a) $y = x^2 + 2$

b) $y = -\frac{1}{3}x^2$

c) $y = x^2$

d) $y = (x - 2)^2$



Practise

For help with questions 1 to 4, see the Investigate.

1. Sketch graphs of these three quadratic relations on the same set of axes.

a) $y = -3x^2$ **b)** $y = \frac{1}{4}x^2$

c) $y = -\frac{1}{4}x^2$

2. Sketch graphs of these three quadratic relations on the same set of axes.

a) $y = (x - 9)^2$ **b)** $y = (x + 2)^2$

c) $y = (x - 5)^2$

3. Sketch graphs of these three quadratic relations on the same set of axes.

a) $y = x^2 + 8$ **b)** $y = x^2 - 5$

c) $y = x^2 - 10$

4. Sketch the graph of each parabola. Label at least three points on the parabola. Describe the transformation from the graph of $y = x^2$.

a) $y = 4x^2$ **b)** $y = \frac{2}{3}x^2$

c) $y = x^2 - 5$ **d)** $y = (x - 8)^2$

e) $y = -\frac{1}{2}x^2$ **f)** $y = (x + 3)^2$

g) $y = x^2 + 0.5$ **h)** $y = -x^2 + 2$

5. a) Make tables of values for $y = x^2$, $y = 2x^2$, $y = x^2 + 1$, and $y = (x - 3)^2$.

b) Compare the y-values for $y = x^2$ and $y = 2x^2$.

c) Compare the y-values for $y = x^2$ and $y = x^2 + 1$.

d) Compare the y-values for $y = x^2$ and $y = (x - 3)^2$.

Connect and Apply

6. Write an equation for the quadratic relation that results from each transformation.

a) The graph of $y = x^2$ is translated 6 units upward.

b) The graph of $y = x^2$ is translated 4 units downward.

7. Write an equation for the quadratic relation that results from each transformation.

a) The graph of $y = x^2$ is translated 7 units to the left.

b) The graph of $y = x^2$ is translated 5 units to the right.

c) The graph of $y = x^2$ is translated 8 units to the left.

d) The graph of $y = x^2$ is translated 3 units to the right.

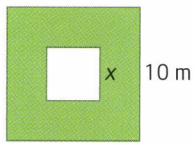
8. Write an equation for the quadratic relation that results from each transformation.

a) The graph of $y = x^2$ is stretched vertically by a factor of 8.

b) The graph of $y = x^2$ is compressed vertically by a factor of $\frac{1}{5}$.

For help with question 9, see the Example.

9. The grass in the backyard of a house is a square with side length 10 m. A square patio is placed in the centre. If the side length, in metres, of the patio is x, then the area of grass remaining is given by the relation $A = -x^2 + 100$.



a) Graph the relation.

b) Find the intercepts. What do they represent?

c) How does the equation change if the grass in the backyard of a house is a square with side length 12 m?

d) For what values of x is each equation valid?

10. The relation $l = 0.04s^2$ can be used to calculate the length, l, in metres, of the skid mark for a car travelling at a speed, s, in kilometres per hour, on dry pavement before braking.

a) What is the length of the skid mark for a car travelling at 50 km/h? 100 km/h?

b) How do the results in part a) compare?

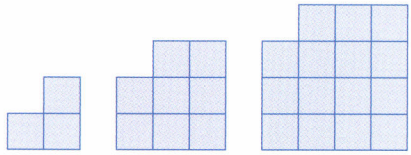
c) For what values of s is this model valid?

d) How would the skid marks and the equation change if the pavement were wet?

Did You Know?

A Technical Collision Investigator or Reconstructionist is a specially trained police officer who investigates serious traffic accidents. These officers collect and interpret evidence to determine the cause of the collision and if any charges should be laid.

11. The first three diagrams in a pattern are shown. Each square has a side length of 1 unit.



a) Make a table comparing base length and area. Use finite differences to determine whether the relation is linear, quadratic, or neither.

b) Determine an equation for the relationship between the base length and the area.

c) Describe the transformation from the graph of $y = x^2$.

Extend

12. The transformations to graph $y = ax^2$ and $y = x^2 + k$ both follow what is indicated by the operation, but in $y = (x - h)^2$, the transformation is opposite to what the operation seems to indicate.

a) Explain why this might be so.

b) Describe the transformation you would use to graph $y = (2x)^2$.

13. A parabola $y = ax^2 + k$ passes through the points $(-1, 3)$ and $(3, -13)$. Find the values of a and k.

14. Compare the graphs of $y = (x - 2)^2$ and $y = (2 - x)^2$. Explain any similarities and differences.

15. Math Contest

a) Identify the similarities and differences in the graphs of $y = (x - 2)^2 + 5$ and $x = (y - 2)^2 + 5$.

b) Solve the second equation for y.

4.4 Graph $y = a(x - h)^2 + k$

The world's most important industry competition for fireworks manufacturers is L'International des Feux Loto-Québec. This event, also known as the Montréal Fireworks Festival, is held each summer in Montréal. The fireworks are synchronized to music that is also broadcast over a local radio station. Competing countries are judged on the synchronization, choice of music, and quality and originality of the visual display.

Paths of projectiles, such as rockets, balls, and fireworks, are often modelled using quadratic relations.

Investigate

How do the graphs of $y = a(x - h)^2 + k$ and $y = x^2$ compare?

Tools

- TI-83 Plus or TI-84 Plus graphing calculator
- grid paper

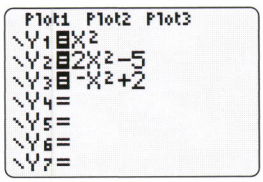
Technology Tip

Turn off all stat plots by pressing 2nd Y= for [STAT PLOT], selecting **4:PlotsOff**, and then pressing ENTER.

1. First, clear any graphed equations, and ensure all stat plots are turned off.

2. Use a standard window.
   - Press ZOOM and select **6:ZStandard.**
   - View the window settings by pressing WINDOW.

3. Graph the equations $y = x^2$, $y = 2x^2 - 5$, and $y = -x^2 + 2$.



4. **a)** Sketch all three graphs on the same set of axes.
   b) Label the coordinates of the vertex and a second point on each parabola.
   c) Describe the transformations.
   d) Without using a graphing calculator, sketch the graph of $y = -2x^2 + 1$.

5. **a)** Clear all equations except $y = x^2$.
   b) Graph the equation $y = (x - 2)^2 + 1$.

6. a) Sketch the two graphs on the same set of axes.

b) Label the coordinates of the vertex and a second point on each parabola.

c) Draw the axis of symmetry for each parabola. Label each axis of symmetry with its equation.

d) Describe the transformations.

7. Without using a graphing calculator, sketch the graph of $y = (x - 1)^2 + 3$.

8. a) Repeat steps 5 and 6 for the equation $y = (x + 5)^2 - 4$.

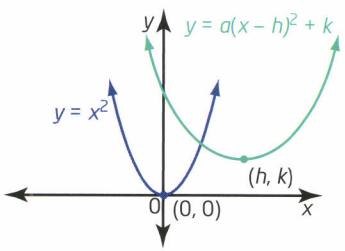
b) Without using a graphing calculator, sketch the graph of $y = (x + 4)^2 + 2$.

9. a) Repeat steps 5 and 6 for the equation $y = 2(x - 1)^2 - 5$.

b) Without using a graphing calculator, sketch the graph of $y = -0.5(x + 2)^2 + 3$.

10. Reflect Write a summary of how to sketch a graph of a quadratic relation of the form $y = a(x - h)^2 + k$. Include a description of how to determine the coordinates of the vertex, the equation of the axis of symmetry, the values that x may take, and the values that y may take.

You can find the following from a quadratic relation of the form $y = a(x - h)^2 + k$:



- The vertex of the parabola is (h, k), representing a horizontal translation of h units and a vertical translation of k units relative to the graph of $y = x^2$.
- The axis of symmetry of the parabola is the vertical line through the vertex with equation $x = h$.
- a indicates the vertical stretch or compression factor relative to the graph of $y = x^2$.
  - If $a > 0$, the parabola opens upward, and the vertex is the minimum point on the graph.
  - If $a < 0$, the parabola opens downward, and the vertex is the maximum point on the graph.

Example 1 Sketch the Graph of $y = a(x - h)^2 + k$

a) Describe the properties of the parabola with equation $y = 2(x - 4)^2 - 3$.

b) Sketch a graph of the parabola and label it fully.

c) Describe the set of values that x may take.

d) Describe the set of values that y may take.

Solution

a) Compare $y = 2(x - 4)^2 - 3$ with $y = a(x - h)^2 - k$.

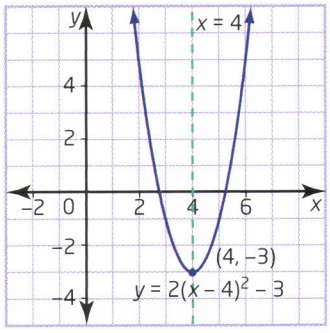
Since $a = 2$, the graph of $y = 2(x - 4)^2 - 3$ will be stretched by a factor of 2 compared to the graph of $y = x^2$.

The parabola will open upward, since a is positive.

The vertex is (h, k), or $(4, -3)$, and it is a minimum point.

The equation of the axis of symmetry is $x = 4$.

b)

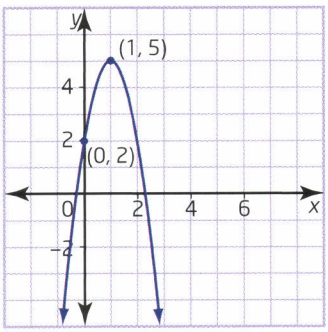


c) The graph shows that x may be any real number.

d) The graph shows that y may be any real number greater than or equal to -3, or $y \geq -3$.

Example 2 Write an Equation for a Graph

Determine an equation for the parabola shown.



Solution

The vertex is (1, 5), so $h = 1$ and $k = 5$.
The parabola opens downward, so a is negative.

Substitute the values for h and k into the equation $y = a(x - h)^2 + k$.

$y = a(x - 1)^2 + 5$

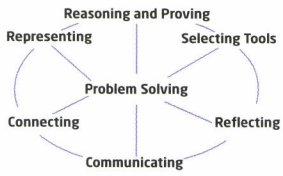
The parabola passes through the point (0, 2). Substitute $x = 0$ and $y = 2$ and solve for a.

$2 = a(0 - 1)^2 + 5$
$2 = a(-1)^2 + 5$
$2 = a + 5$
$a = -3$

An equation for the parabola is $y = -3(x - 1)^2 + 5$.

Example 3 Fireworks

At a fireworks display, a firework is launched from a height of 2 m above the ground and reaches a maximum height of 40 m at a horizontal distance of 10 m.



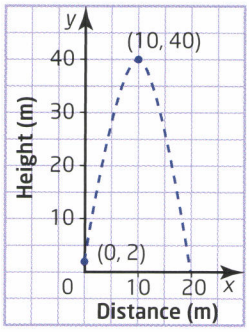
a) Determine an equation to model the flight path of the firework.

b) The firework continues to travel an additional 1 m horizontally, after it reaches its maximum height, before it explodes. What is its height when it explodes?

c) At what other horizontal distance is the firework at the same height as in part b)?

Solution

a) Sketch a graph of the situation.



The launch height of 2 m above the ground represents the point (0, 2).

The maximum height is 40 m at a horizontal distance of 10 m, so the vertex is (10, 40).
Substitute $h = 10$ and $k = 40$ into the equation $y = a(x - h)^2 + k$.

$y = a(x - 10)^2 + 40$

Substitute $x = 0$ and $y = 2$ and solve for a.

$2 = a(0 - 10)^2 + 40$
$2 = a(-10)^2 + 40$
$2 = 100a + 40$
$-38 = 100a$
$a = -\frac{38}{100}$
$a = -0.38$

An equation that models the flight path of the firework is $y = -0.38(x - 10)^2 + 40$, where x is the horizontal distance travelled, in metres, after the firework is launched and y is the height, in metres, above the ground.

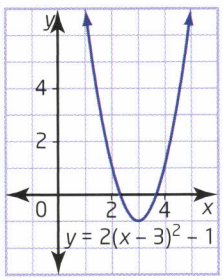
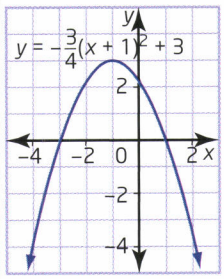
b) The firework travels an additional 1 m horizontally after its maximum height of 40 m at $x = 10$, so $x = 11$. Substitute $x = 11$ into $y = -0.38(x - 10)^2 + 40$.

$$\begin{aligned} y &= -0.38(11 - 10)^2 + 40 \\ &= -0.38(1)^2 + 40 \\ &= 39.62 \end{aligned}$$

The firework exploded at a height of 39.62 m.

c) Due to the symmetric property of a parabola, the firework is at the same height 1 m before the maximum point, or at a horizontal distance of 9 m.

Key Concepts

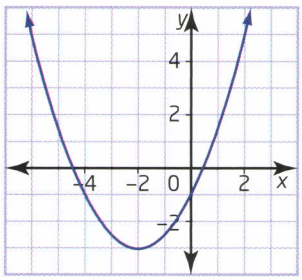
Property	$y = a(x - h)^2 + k$	$y = 2(x - 3)^2 - 1$	$y = -\frac{3}{4}(x + 1)^2 + 3$
Vertex	(h, k)	$(3, -1)$	$(-1, 3)$
Axis of symmetry	$x = h$	$x = 3$	$x = -1$
Stretch or compression factor relative to $y = x^2$	a	2	$-\frac{3}{4}$
Direction of opening	If $a > 0$, the parabola opens upward. The vertex is a minimum point. If $a < 0$, the parabola opens downward. The vertex is a maximum point.	Upward. $(3, -1)$ is a minimum point.	Downward. $(-1, 3)$ is a maximum point.
Graph	Parabola		
Values x may take	Any real number. Also depends on the situation.	Set of real numbers.	Set of real numbers.
Values y may take	If $a > 0$, then $y \geq k$. If $a < 0$, then $y \leq k$. Also depends on the situation.	$y \geq -1$	$y \leq 3$

Communicate Your Understanding

C1 Why is the vertical line through the vertex called the axis of symmetry? Illustrate with an example.

C2 When describing the transformation from $y = x^2$ to $y = 2x^2$, you say that it has been stretched vertically by a factor of 2, instead of compressed horizontally. Explain why vertical stretches are used in descriptions.

C3 Which equation is correct for the graph shown? Explain your reasoning.



A $y = (x + 2)^2 - 3$

B $y = \frac{1}{3}(x + 2)^2 - 3$

C $y = \frac{1}{2}(x + 2)^2 - 3$

D $y = -2(x + 2)^2 - 3$

Practise

For help with questions 1 and 2, see Example 1.

1. Copy and complete the table for each parabola. Replace the heading for the second column with the equation for the parabola.

Property	$y = a(x - h)^2 + k$
Vertex	
Axis of symmetry	
Stretch or compression factor relative to $y = x^2$	
Direction of opening	
Values x may take	
Values y may take	

a) $y = (x - 4)^2$

b) $y = (x - 2)^2 - 4$

c) $y = (x + 3)^2 - 2$

d) $y = \frac{1}{2}(x + 1)^2 + 5$

e) $y = (x - 7)^2 - 3$

f) $y = -(x - 1)^2 + 7$

g) $y = 2(x - 4)^2 - 5$

h) $y = -3(x + 4)^2 - 2$

2. Sketch each parabola in question 1.

For help with questions 3 to 7, see Example 2.

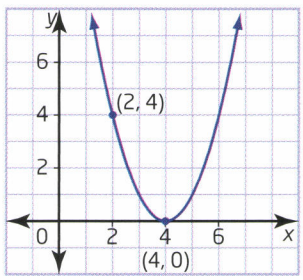
3. Write an equation for the parabola with vertex at (2, 3), opening upward, and with no vertical stretch.

4. Write an equation for the parabola with vertex at (−3, 0), opening downward, and with a vertical stretch of factor 2.

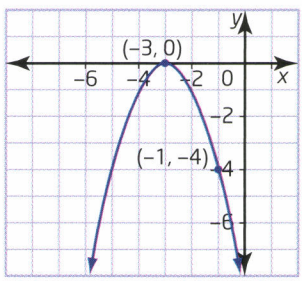
5. Write an equation for the parabola with vertex at (4, −1), opening upward, and with a vertical compression of factor 0.3.

6. Write an equation for each parabola.

a)

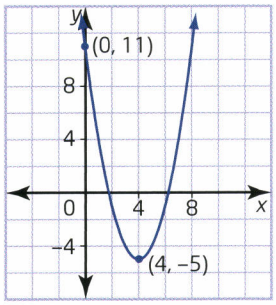


b)

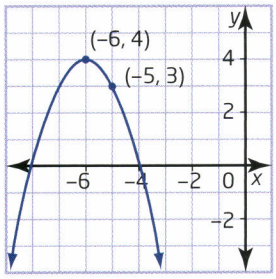


7. Write an equation for each parabola.

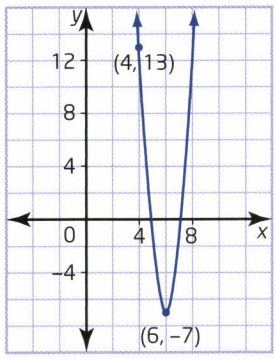
a)



b)



c)



Connect and Apply

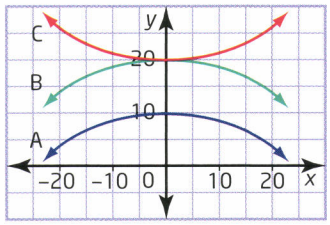
8. The graph of $y = x^2$ is stretched vertically by a factor of 3 and then translated 2 units to the left and 1 unit down. Sketch the parabola and write its equation.

9. The graph of $y = x^2$ is reflected in the x-axis, compressed vertically by a factor of $\frac{1}{2}$, and then translated 2 units upward. Sketch the parabola and write its equation.

10. a) Find an equation for the parabola with vertex (1, 4) that passes through the point (3, 8).

b) Find an equation for the parabola with vertex (−2, 5) and y-intercept 1.

11. A stadium roof has a cross section in the shape of a parabolic arch with equation $y = -\frac{1}{45}x^2 + 20$. Which graph represents the arch? Justify your reasoning.



For help with questions 12 and 13, see Example 3.

12. The path of a soccer ball is modelled by the relation $h = -\frac{1}{16}(d - 28)^2 + 49$, where d is the horizontal distance, in metres, after it was kicked, and h is the height, in metres, above the ground.

a) Sketch the path of the soccer ball.

b) What is the maximum height of the ball?

c) What is the horizontal distance when this occurs?

d) What is the height of the ball at a horizontal distance of 20 m?

e) Find another horizontal distance where the height is the same as in part d).

13. A baseball is batted at a height of 1 m above the ground and reaches a maximum height of 33 m at a horizontal distance of 4 m.

a) Determine an equation to model the path of the baseball.

b) What is the height of the baseball once it has travelled a horizontal distance of 6 m?

c) At what other horizontal distance is the baseball at the same height as in part b)?

14. The flight path of a firework is modelled by the relation $h = -5(t - 5)^2 + 127$, where h is the height, in metres, of the firework above the ground and t is the time, in seconds, since the firework was fired.

a) What is the maximum height reached by the firework? How many seconds after it was fired does the firework reach this height?

b) How high was the firework above the ground when it was fired?

15. Parabolic mirrors are used in telescopes to help magnify the image. The cross section of a parabolic mirror is shown.

a) Sketch a graph to represent the cross section of the mirror, placing the vertex at $(0, -0.24)$.

b) Write an equation to represent the cross section of the mirror. Describe the values of x for which your equation is valid.

c) Move the mirror to a different location on the same set of axes. Write a different equation to represent the cross section of the mirror. Describe the values of x for which your equation is valid.

Did You Know?

The Olympic Torch is lit by placing it in a parabolic mirror that concentrates rays of sunlight.

16. Chapter Problem Use Technology A city opened a new landfill site in 2000. In Section 4.1, question 6, you created a table of values showing the total mass of garbage in the landfill for each year from 2000 to 2007.

a) Use a graphing calculator to create a scatter plot of the data and draw a curve of best fit.

- With the scatter plot displayed, press STAT, cursor over to display the **CALC** menu, and select **5:QuadReg**.
- Press VARS, and cursor over to display the **Y-VARS** menu. Select **1:Function** and then select **1:Y1**.
- Press ENTER to get the **QuadReg** screen, and press GRAPH.

b) Use the Minimum operation of a graphing calculator to find the coordinates of the vertex.

- Adjust the window settings so you can view the vertex of the parabola.
- Press 2nd [CALC] to display the **CALCULATE** menu, and select **3:minimum**.
- Move the cursor to the left of the vertex and press ENTER.
- Move the cursor to the right of the vertex and press ENTER.
- Press ENTER again.

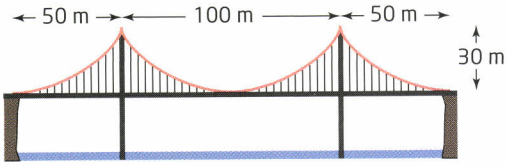
c) Sketch the graph of the curve of best fit. Label the coordinates of the vertex and one other point on your sketch. Use these points to determine an equation for the curve of best fit in the form $y = a(x - h)^2 + k$.

d) Describe the set of values that x may take and the set of values that y may take.

Making Connections

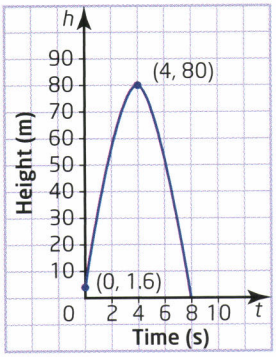
Refer to the Technology Appendix for help with creating a scatter plot on a TI-83 Plus or TI-84 Plus graphing calculator.

17. The cables of a suspension bridge form parabolas. If the minimum point of the centre cable is placed at the origin, determine an equation for each parabola. Describe the values of x for which each equation is valid.



Achievement Check

18. The graph shows the path of a rocket fired from the deck of a barge in Lake Ontario at a Canada Day fireworks display. It is a parabola, where h is the height, in metres, of the rocket above the water and t is the time, in seconds.



a) What is the maximum height reached by the rocket? Justify your answer.

b) When did the rocket reach its maximum height? Justify your answer.

c) How high was the rocket above the water when it was set off? Explain your answer.

d) Find an equation to describe the flight of the rocket.

e) After how long did the rocket fall into the water? Explain your answer.

Extend

19. A parabola has equation $y = 2(x - 4)^2 - 1$. Write an equation for the parabola after each transformation.

a) a reflection in the x-axis

b) a translation of 4 units to the left

c) a reflection in the x-axis, followed by a translation of 3 units upward

d) a reflection in the y-axis

20. a) In Chapter 2, you developed the equation of a circle centred at the origin. What is the equation of each circle?

- radius 5, centred at (0, 3)
- radius 7, centred at (6, 1)
- radius 8, centred at (−3, 5)
- radius r, centred at (h, k)

b) What do the equation of a circle and the equation of a parabola have in common?

21. Math Contest A locus is a set of points that satisfy a specific condition. For example, a circle is a set of points that are equidistant from a fixed point (the centre). Find an equation for the locus of points that is equidistant from the point (3, 2) and the line $y = -5$.

22. Math Contest Given that $a + b = 21$ and $\frac{1}{a} + \frac{1}{b} = \frac{7}{18}$, the value of ab is

A 18

B 36

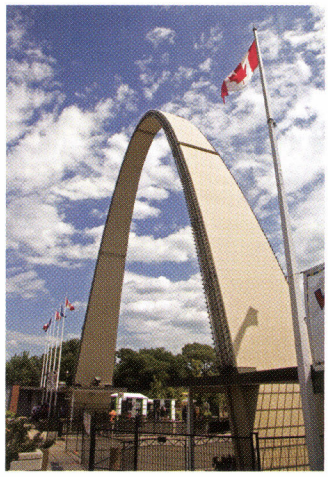
C 54

D 72

E 90

4.5 Quadratic Relations of the Form $y = a(x - r)(x - s)$

The Dufferin Gate at the west end of the Canadian National Exhibition in Toronto is a parabolic arch. The arch is approximately 20 m high and approximately 22 m wide. In this section, you will model the arch using an equation of the form $y = a(x - r)(x - s)$.



Investigate

Tools
- grid paper

How does the equation $y = a(x - r)(x - s)$ relate to its graph?

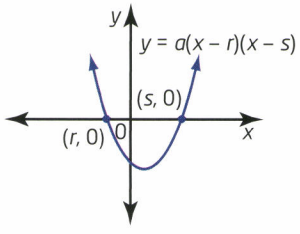
1. Create a table of values for the equation $y = 2(x - 5)(x + 3)$. Use integer values of x from -6 to 6.
2. Use finite differences to show that this is a quadratic relation.
3. Graph the relation.
4. Identify the x-intercepts. How do they relate to the equation?
5. Identify the coordinates of the vertex. How could you use the x-intercepts to find the coordinates of the vertex?
6. **Reflect** Summarize your findings. Describe how to determine the x-intercepts and vertex of $y = a(x - r)(x - s)$.
7. **a)** Use your method to analyse the graph of $y = -\frac{1}{2}(x + 2)(x - 6)$.

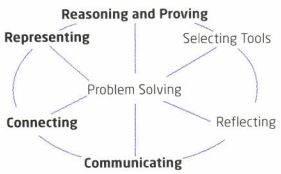
   b) Sketch the graph of $y = -\frac{1}{2}(x + 2)(x - 6)$.

Making Connections

You will learn how to convert between the forms $y = ax^2 + bx + c$ and $y = a(x - r)(x - s)$ in Chapter 5.

A relation of the form $y = a(x - r)(x - s)$ is quadratic. The x-intercepts, or zeros, are r and s.





Making Connections

In Chapter 2, you found the midpoint between two points.

Example 1 Graph of $y = a(x - r)(x - s)$

Describe the graph of the quadratic relation $y = 2(x + 1)(x - 7)$. Sketch the graph and label the x-intercepts, vertex, and axis of symmetry.

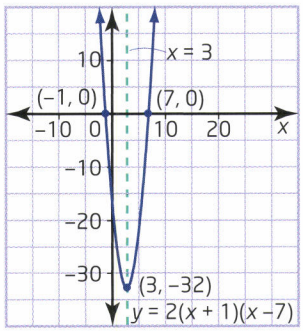
Solution

Compare $y = 2(x + 1)(x - 7)$ with $y = a(x - r)(x - s)$.

The x-intercepts are -1 and 7.

Due to the symmetric property of a parabola, the x-coordinate of the vertex is on the axis of symmetry, which passes through the midpoint of the line segment connecting the x-intercepts.

$$x = \frac{-1 + 7}{2}$$
$$= 3$$



To find the y-coordinate of the vertex, substitute $x = 3$ into the equation.

$$y = 2(x + 1)(x - 7)$$
$$= 2(3 + 1)(3 - 7)$$
$$= 2(4)(-4)$$
$$= -32$$

The vertex is $(3, -32)$.

The axis of symmetry has equation $x = 3$.

Example 2 Dufferin Gate

The Dufferin Gate is a parabolic arch that is approximately 20 m tall and approximately 22 m wide.

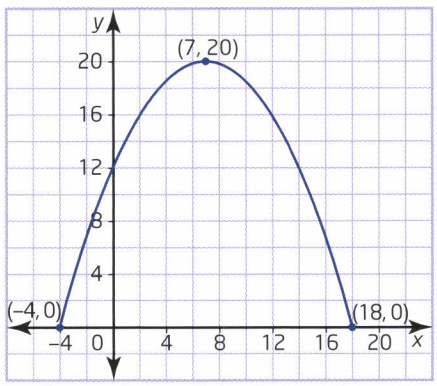
a) Sketch a graph of the arch with the left base located on the x-axis 4 units to the left of the y-axis. Label the x-intercepts and vertex.

b) Determine an equation to model the arch.

Solution

a) The x-intercepts are -4 and 18, so the x-coordinate of the vertex is $\frac{-4 + 18}{2}$, or 7.

The maximum height is 20 m, so the vertex is (7, 20).



b) Use the x-intercepts to determine that the equation has the form $y = a(x + 4)(x - 18)$.

Substitute the coordinates of the vertex and solve for a.

$20 = a(7 + 4)(7 - 18)$

$20 = a(11)(-11)$

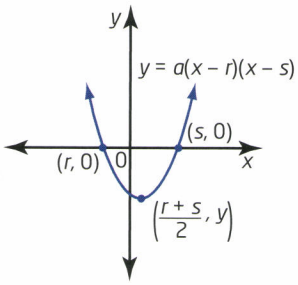
$20 = -121a$

$a = -\frac{20}{121}$

An equation representing the shape of the arch is $y = -\frac{20}{121}(x + 4)(x - 18)$, where y is the height, in metres, and x is the horizontal distance, in metres.

Key Concepts

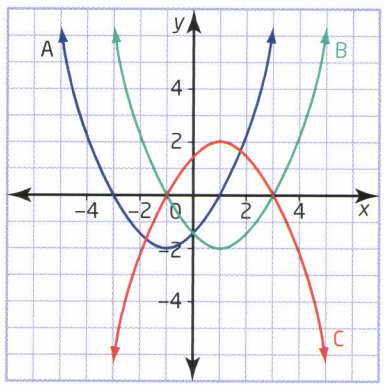
- A relation of the form $y = a(x - r)(x - s)$ is quadratic.
- The x-intercepts, or zeros, are r and s.
- The x-coordinate of the vertex is on the axis of symmetry, which passes through the midpoint of the line connecting the x-intercepts.



Communicate Your Understanding

C1 With the help of an example, describe a process that you can use to find the vertex of a parabola with equation of the form $y = a(x - r)(x - s)$.

C2 Which is the correct graph of the quadratic relation $y = 0.5(x - 3)(x + 1)$? Explain how you know.



Practise

For help with questions 1 to 4, see Example 1.

1. Sketch graphs of all three relations on the same set of axes. Label the x-intercepts, vertex, and axis of symmetry for each parabola. Then, describe the similarities and differences between the graphs.

a) $y = (x + 3)(x - 1)$

b) $y = 2(x + 3)(x - 1)$

c) $y = -2(x + 3)(x - 1)$

2. Sketch graphs of all three relations on the same set of axes. Label the x-intercepts, vertex, and axis of symmetry for each parabola. Then, describe the similarities and differences between the graphs.

a) $y = (x - 4)(x - 8)$

b) $y = \frac{1}{2}(x - 4)(x - 8)$

c) $y = \frac{1}{4}(x - 4)(x - 8)$

3. Sketch each parabola. Label the x-intercepts and vertex.

a) $y = (x - 6)(x - 2)$

b) $y = -(x + 3)(x + 7)$

c) $y = 2(x - 3)(x + 2)$

d) $y = -2(x - 4)(x + 2)$

4. Sketch each parabola. Label the x-intercepts and vertex.

a) $y = 3x(x + 2)$

b) $y = \left(x + \frac{1}{2}\right)\left(x - \frac{7}{4}\right)$

c) $y = -0.2(x - 4)(x + 10)$

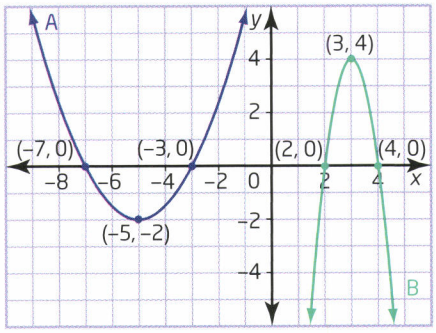
d) $y = \frac{2}{3}(x - 6)(x + 9)$

e) $y = (x + 3.5)(x - 3.5)$

f) $y = -\frac{1}{3}(x - 0.5)(x - 0.1)$

For help with question 5, see Example 2.

5. Determine an equation in the form $y = a(x - r)(x - s)$ to represent each parabola. Consider the vertex and x-intercepts.



Connect and Apply

6. You investigated the graphs of $y = (x - h)^2$ in Section 4.3. Consider the quadratic relation $y = (x - 5)^2$.

a) Write the coordinates of the vertex of the parabola.

b) How many x-intercepts does the parabola have?

c) Rewrite the equation in the form $y = a(x - r)(x - s)$.

7. A parabola has equation $y = (x + 2)^2$.

a) Write its x-intercepts.

b) Determine the coordinates of its vertex.

8. The predicted flight path of a toy rocket is defined by the relation $h = -2(d - 3)(d - 15)$, where d is the horizontal distance, in metres, from a safety wall, and h is the height, in metres, above the ground.

a) Sketch a graph of the path of the rocket.

b) How far from the wall is the rocket when it lands on the ground?

c) What is the maximum height of the rocket, and how far, horizontally, is it from the wall at that moment?

9. Investigate the list of quadratic relations and make a conjecture relating the x-intercepts and the coordinates of the vertex. Test your conjecture with another example.

$y = (x + 1)(x - 1)$
$y = (x - 4)(x + 4)$
$y = -(x + 7)(x - 7)$
$y = 2(x + 6)(x - 6)$
$y = -\frac{1}{2}(x - 5)(x + 5)$

10. A soccer ball is kicked from a point 23 m to the left of the halfway line and lands at a point 17 m to the right of the halfway line. It reaches a maximum height of 10 m during its parabolic flight.

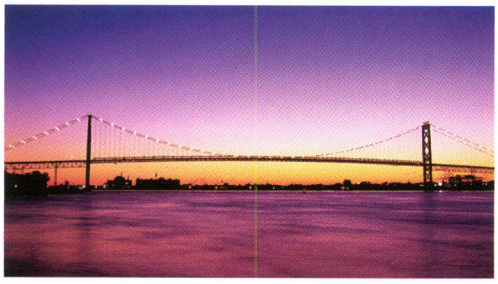
a) Sketch a graph to show the flight of the soccer ball if the halfway line passes through the origin.

b) Determine an equation to represent the path of the soccer ball.

11. The Ambassador Bridge is a suspension bridge that crosses the Detroit River and connects Windsor, Ontario, to Detroit, Michigan. The two towers that support the centre span of cables rise 118 m above the river and are 564 m apart. The cable reaches its lowest point approximately 46 m above the river.

a) Sketch a graph to show the curve of the cable if the origin is centred under the lowest point of the cable at the river's surface.

b) Determine an equation to represent the curve of the cables in the form $y = a(x - r)(x - s)$, if possible. If not, explain why.



12. An architect designed the cross section of a museum using the quadratic relations $y = -\frac{1}{2}(x - 2)(x + 10)$, $y = -(x + 4)(x - 4)$, and $y = -1.5(x - 2)(x - 10)$, where y is the height, in metres, of the building, and x is the horizontal distance, in metres, from the middle of the building.

a) Sketch a graph of the cross section of the museum.

b) How wide is the entire museum?

c) How tall is each section of the museum?

13. The revenue, R, from concert ticket sales at a local venue is calculated as (number of tickets sold) $\times$ (price of ticket). The current price of a ticket is \$20, and the venue typically sells 100 tickets. For each \$1 increase in ticket price, 10 fewer tickets are sold. So, the revenue can be modelled using the equation $R = (100 - 10x)(20 + x)$, where x represents the number of \$1 increases.

a) Rewrite the equation in the form $R = a(x - r)(x - s)$.

b) Sketch a graph of the relation.

c) What does the R-intercept represent? What do the x-intercepts represent?

d) What does a negative value of x represent?

e) What price maximizes the revenue?

Extend

14. **Use Technology** Use a graphing calculator to graph each relation. Describe and justify how the shape of the graph relates to the number of factors.

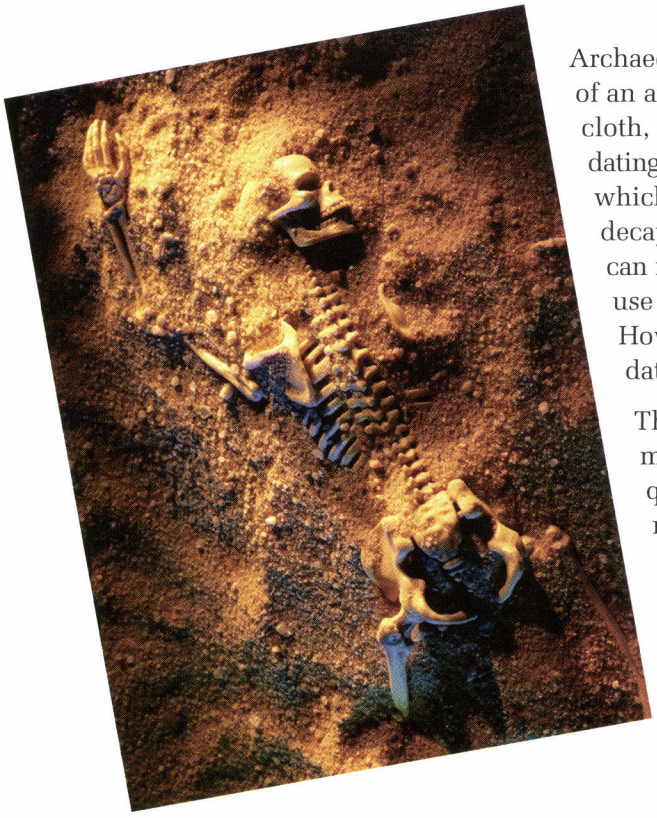
a) $y = (x - 2)(x - 5)(x - 7)$

b) $y = 2(x + 6)(x + 3)(x - 1)(x - 9)$

c) $y = -x(x + 2)(x - 4)(x - 6)(x - 9)$

15. A point lies 2 units above the x-axis on the axis of symmetry of the parabola with equation $y = (x + 4)(x - 6)$. How far is it from this point to the point on the curve where $x = 5$?

4.6 Negative and Zero Exponents



Archaeologists use radioactivity to determine the age of an artifact. For organic remains, such as bone, cloth, and wood, the typical method used is carbon-14 dating. This method focuses on the decay of carbon-14, which begins when an organism dies. Since this decay occurs at a constant, known rate, scientists can measure the amount of carbon-14 remaining and use a formula to determine the age of the sample. However, carbon-14 dating is only reliable for dating artifacts up to about 50 000 years old.

The decay of radioactive elements can be modelled mathematically, but not with a quadratic relation—scientists use a different non-linear model called an exponential relation.

Investigate

How can you determine the meaning of negative and zero exponents?

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

A: Compare $y = x^2$ and $y = 2^x$

Compare the quadratic relation $y = x^2$ to the exponential relation $y = 2^x$.

1. Enter the window settings by pressing WINDOW and changing the values to match those shown. Enter the equation $y = x^2$ as **Y1** and the equation $y = 2^x$ as **Y2**. Press GRAPH.

```
WINDOW
Xmin=-10
Xmax=10
Xscl=1
Ymin=-2
Ymax=50
Yscl=5
Xres=1
```

2. How is the graph of $y = 2^x$ similar to the graph of $y = x^2$? How is it different?

3. Press 2nd [TABLE] to see a table of values for each relation. Compare the results for integer values of x from -3 to 3. Identify which relation grows faster over various intervals of x.

4. Make a table of values for $y = 2^x$, using integer values of x from -3 to 3. Express y-values in fraction form in lowest terms.

Technology Tip

You can change a decimal to a fraction.

- Enter the decimal value.
- Press MATH, select **1:▶Frac**, and then press ENTER.

5. Compare the values of 2^3 and 2^{-3}, 2^2 and 2^{-2}, and 2^1 and 2^{-1}. What do you notice?

6. What is the value of 2^0?

7. **Reflect** How does the exponential relation $y = 2^x$ help you understand the meaning of negative and zero exponents?

B: Use Patterns

Tools
- grid paper

1. **a)** Copy and complete the list of decreasing powers of 2.

$2^5 = 32$
$2^4 = 16$
$2^3 =$ ■
$2^2 =$ ■
$2^1 =$ ■

b) As you move down the list, by what fraction would you multiply each power to get the next result?

c) Extend the list. Use the pattern to determine the value of each power of 2.

$2^0 =$ ■
$2^{-1} =$ ■
$2^{-2} =$ ■
$2^{-3} =$ ■
$2^{-4} =$ ■
$2^{-5} =$ ■

2. **a)** Use your results to make a table of values for the relation $y = 2^x$, using integer values of x from -5 to 5.

b) Plot the ordered pairs.

c) Describe the graph. Will it cross the x-axis? If so, at what value? If not, why not?

d) Draw a curve of best fit.

3. Repeat steps 1 and 2 for powers of 3.

4. **Reflect** If a is any non-zero base, summarize your findings by describing how to evaluate the following.

a) a^0

b) a^{-1}

c) a^{-2}

d) a^{-3}

C: Use Exponent Laws

1. a) Copy and complete the table by simplifying each expression twice. First, expand and divide, and then use the exponent law for division.

Expression	Expand and Divide	Exponent Law
$\frac{3^2}{3^5}$	$\frac{\cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times 3 \times 3 \times 3} = \frac{1}{3^3}$	$\frac{3^2}{3^5} = 3^{2-5}$ $= 3^{-3}$
$\frac{4^1}{4^3}$		
$\frac{2^4}{2^7}$		
$\frac{(-5)^2}{(-5)^3}$		
$\frac{(-2)^3}{(-2)^5}$		

b) How do the two results in each row compare? Does the sign of the base make a difference?

2. a) Copy and complete the table by simplifying each expression twice. First, expand and divide, and then use the exponent law for division.

Expression	Expand and Divide	Exponent Law
$\frac{3^5}{3^5}$	$\frac{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}}{\cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3} \times \cancel{3}} = 1$	$\frac{3^5}{3^5} = 3^{5-5}$ $= 3^0$
$\frac{5^2}{5^2}$		
$\frac{4^3}{4^3}$		
$\frac{(-3)^4}{(-3)^4}$		
$\frac{(-2)^2}{(-2)^2}$		

b) How do the two results in each row compare? Does the sign of the base make a difference?

3. Reflect

a) Write a rule for a base raised to a negative exponent.

b) Write a rule for a base raised to the exponent 0.

4. If a is any non-zero base, use your rules to evaluate each power.

a) a^0

b) a^{-1}

c) a^{-2}

d) a^{-3}

When a base is raised to a negative exponent, it is equal to the reciprocal of the base raised to the positive of the exponent. For example,

$$2^{-3} = \frac{1}{2^3}$$
$$= \frac{1}{8}$$

When a base is raised to an exponent of 0, the result is 1. For example, $2^0 = 1$.

Example 1 Negative and Zero Exponents

Evaluate.

a) 4^{-1} **b)** 8^0 **c)** $(-2)^{-3}$ **d)** $\left(\frac{2}{3}\right)^{-2}$ **e)** 0^{-2}

Solution

a) $4^{-1} = \frac{1}{4^1}$

$= \frac{1}{4}$

b) $8^0 = 1$

c) $(-2)^{-3} = \frac{1}{(-2)^3}$

$= \frac{1}{(-2)(-2)(-2)}$

$= -\frac{1}{8}$

d) $\left(\frac{2}{3}\right)^{-2} = \frac{1}{\left(\frac{2}{3}\right)^2}$

$= \frac{1}{\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)}$

$= \frac{1}{\frac{4}{9}}$

$= 1 \times \frac{9}{4}$

$= \frac{9}{4}$

e) 0^{-2} is undefined because it has denominator 0 when written as $\frac{1}{0^2}$.

Example 2 Radioactive Decay

Did You Know?

The time it takes for a radioactive element to decay to half its original amount is called its *half-life*. The half-life of carbon-14 is 5700 years.

Carbon-14 is a radioactive element that decays to $\frac{1}{2}$, or 2^{-1}, of its original amount after every 5700 years. Determine the remaining mass of 10 g of carbon-14 after

a) 11 400 years

b) 28 500 years

Solution

a) For each 5700 years, the original amount decreases by a factor of $\frac{1}{2}$, or 2^{-1}.

Find how many 5700-year periods of time are in 11 400 years.

11 400 years $= 2 \times 5700$ years

$$\left(\frac{1}{2}\right)^2 \times 10 = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \times 10$$
$$= \frac{1}{4} \times 10$$
$$= 2.5$$

or

$$(2^{-1})^2 \times 10 = 2^{-1 \times 2} \times 10$$
$$= 2^{-2} \times 10$$
$$= \frac{1}{2^2} \times 10$$
$$= \frac{1}{4} \times 10$$
$$= 2.5$$

Multiplying by $\frac{1}{2}$ or 2^{-1} is the same as dividing by 2. So, I can also find the answer by dividing by 2, twice.

$10 \div 2 \div 2 = 5 \div 2$
$= 2.5$

The remaining mass after 11 400 years is 2.5 g.

b) 28 500 years $= 4 \times 5700$ years

$$(2^{-1})^4 \times 10 = 2^{-4} \times 10$$
$$= \frac{1}{2^4} \times 10$$
$$= \frac{1}{16} \times 10$$
$$= 0.625$$

The remaining mass after 28 500 years is 0.625 g.

Key Concepts

- When a non-zero base is raised to a negative exponent, the result is the reciprocal of the base raised to the positive of the exponent.

  For example, $2^{-3} = \frac{1}{2^3}$.

- When a non-zero number is raised to the exponent 0, the result is 1. For example,

  $1 = \frac{100}{100} = \frac{10^2}{10^2} = 10^{2-2} = 10^0$

Communicate Your Understanding

C1 Explain why 2^{-3} is not a negative number.

C2 Explain why 5^0 has a value of 1.

C3 Explain why it is often better not to rely on a calculator to evaluate powers with a fractional base, such as $\left(\frac{7}{2}\right)^{-2}$.

Practise

For help with questions 1 to 3, see Example 1.

1. Rewrite each power with a positive exponent.

a) 3^{-2} **b)** 5^{-1} **c)** 10^{-4}

d) 7^{-3} **e)** $(-2)^{-4}$ **f)** $(-7)^{-1}$

2. Evaluate.

a) 6^{-2} **b)** 9^0 **c)** 7^{-1}

d) 10^{-3} **e)** $(-9)^{-1}$ **f)** $(-12)^{-2}$

g) $(-3)^0$ **h)** -89^0

3. Evaluate.

a) $\left(\frac{1}{3}\right)^{-2}$ **b)** 0^{-5} **c)** $\left(-\frac{1}{4}\right)^{-1}$

d) $\left(\frac{5}{6}\right)^{-2}$ **e)** $\left(-\frac{3}{8}\right)^{-4}$ **f)** $\left(\frac{9}{4}\right)^{-3}$

Connect and Apply

4. Evaluate using pencil and paper. Check your results using a calculator.

a) $6^0 + 6^{-2}$ **b)** $8 - 8^{-1}$

c) $(4 + 3)^0$ **d)** $4^0 + 3^0$

For help with questions 5 and 6, see Example 2.

5. Iodine-123 is a radioactive element used in medical imaging. It decays to $\frac{1}{2}$, or 2^{-1}, of its original mass after 13 h. After 26 h, it decays to $\frac{1}{4}$, or 2^{-2}, of its original mass.

a) What fraction remains after 52 h?

b) What fraction remains after 78 h?

c) Write each fraction as a power of 2 with a negative exponent.

6. Uranium-238 is a radioactive element found in rocks and many types of soils. Uranium-238 decays to $\frac{1}{2}$, or 2^{-1}, of its original amount after every 4.5 billion years. Determine the remaining mass of 0.5 kg of uranium-238 after

a) 9 billion years

b) 22.5 billion years

7. Radium-226 is a radioactive element that is used in a form of radiation treatment for various types of cancer. Radium-226 decays to $\frac{1}{16}$ of its mass in 6400 years.

a) Write the fraction $\frac{1}{16}$ as a power of 2.

b) What is the remaining mass of 8 mg of radium-226 after 6400 years?

8. Determine the value of x that makes each statement true.

a) $x^{-3} = \frac{1}{27}$ **b)** $x^{-1} = \frac{4}{5}$

c) $2^x = \frac{1}{4}$ **d)** $\left(\frac{2}{5}\right)^x = \frac{125}{8}$

9. Use a pattern, similar to that in Investigate, Part B, to verify that $(-4)^{-2} = \frac{1}{(-4)^2}$.

10. Refer to question 9. Make up your own patterning example to illustrate the meaning of the exponents 0 and -3.

11. The number of bees in a hive is 1000 on June 1 and doubles every month. This can be expressed as $N = 1000 \times 2^t$, where N represents the number of bees and t represents time, in months.

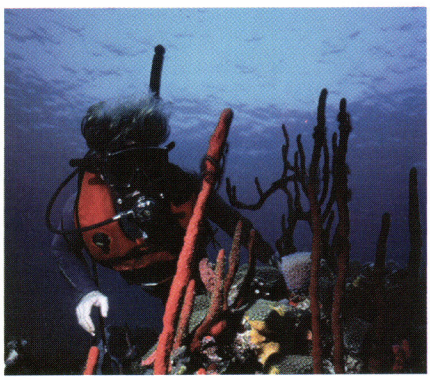
a) Find the number of bees after 2, 3, 4, and 5 months.

b) What does $t = 0$ represent in this situation?

c) Is it possible for t to be -1? What does this mean?

d) When were there 125 bees? Explain.

12. The intensity of light energy under water decreases rapidly. Many factors affect how quickly the intensity decreases. The light energy under water can be calculated using an exponential relation. For example,
Ocean: $I = 325 \times (1.024)^{-d}$
Lake Erie: $I = 401 \times (1.222)^{-d}$
In these relations, d is the depth, in metres, and I is the light energy, in langleys.



a) Why is a negative exponent used in the formulas?

b) Sketch a graph of each relation.

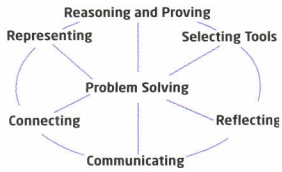
c) In which body of water does the intensity decrease more quickly? Explain why.

Did You Know?

The colours in light are absorbed at different rates, with the red end of the spectrum going first. As the diver descends, everything ends up as shades of blue.

Achievement Check

13. Mitosis is a process of cell reproduction in which one cell divides into two identical cells. A bacterium called *E. coli* often causes serious food poisoning. It can reproduce itself in 15 min.



a) Starting with one bacterium, make a table with time in $\frac{1}{4}$-h blocks and the corresponding number of *E. coli* bacteria up to 2 h of bacteria growth.

b) Make a scatter plot from the table of data.

c) What conclusions can you make from examining the table and scatter plot?

d) About how long will it take before there are 10 000 bacteria?

Extend

14. Chris walks halfway along a 100-m track in 1 min, then half of the remaining distance in the next minute, then half of the remaining distance in the third minute, and so on.

a) How far has Chris walked after 10 min?

b) Will Chris get to the end of the track? Explain. Include a table of values and a graph to support your explanation.

c) Write an equation to model this situation.

Did You Know?

The ancient Greek philosopher Zeno (450 BCE) argued that a number of truths about space and time were false. One of Zeno's paradoxes is that an object can never reach its target since the object must first cover an infinite number of finite distances, and this takes an infinite amount of time. For example, in question 15, Chris covers half of the remaining distance in each minute, so in theory he will never reach the end of the track.

15. When a patient takes a certain drug, $\frac{1}{10}$ of the drug that remains in his or her system is used per hour. A patient is given a 500-mg dose of a drug.

a) Write an equation relating time and the remaining mass of the drug.

b) After how many hours will less than 1% of the original mass remain?

16. Use Technology The popularity of fads and fashions often decays exponentially. One example is ticket sales for a popular movie. The table shows the total money spent per weekend on tickets in the United States and Canada for the movie *The Da Vinci Code*.

Weekend in 2006	Ticket Sales ($millions)
May 19–May 21	77.1
May 26–May 28	34.0
June 2–June 4	18.6
June 9–June 11	10.4
June 16–June 18	5.3
June 23–June 25	4.1
June 30–July 2	2.3

a) Use a graphing calculator to create a scatter plot of the data.

b) Draw a quadratic curve of best fit.
- Press STAT, cursor over to display the **CALC** menu, and select **5:QuadReg**.
- Press VARS, and cursor over to display the **Y-VARS** menu. Select **1:Function** and then select **1:Y1**.
- Press ENTER to get the **QuadReg** screen, and press GRAPH.

c) Draw an exponential curve of best fit.
- Press STAT, cursor over to display the **CALC** menu, and select **0:ExpReg**.
- Press VARS, and cursor over to display the **Y-VARS** menu. Select **1:Function** and then select **2:Y2**.
- Press ENTER to get the **ExpReg** screen, and press GRAPH.

d) Examine the two curves. Which curve of best fit best models the data?

17. Use Technology The table shows the atmospheric pressure compared to the altitude in a particular location.

Altitude (km)	Pressure (millibars)
0	1013.3
1	898.8
2	795.0
3	701.3
4	616.5
5	540.5
6	472.2
7	411.1
8	356.5
9	307.9
10	265.0

a) Make a scatter plot of the data using a graphing calculator.

b) Use the ExpReg operation to determine an exponential curve of best fit.

c) Explain why an exponential model is better than a quadratic one.

18. Sketch the graphs of $y = x^2 + 1$ and $y = \frac{2^x - 2^{-x}}{2}$. Compare the values of y over various intervals.

19. Math Contest Solve each equation for x.

a) $3^x = \frac{1}{81}$

b) $4(2^{3x}) = \frac{1}{16}$

20. Math Contest The integers -3, -2, -1, 0, 1, 2, and 3 are substituted into the expression $a^b + c^d + e^f + g$.

a) What is the greatest possible value of the expression?

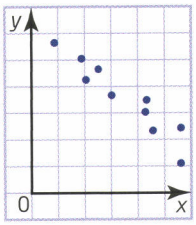
b) What is the least possible value of the expression?

Chapter 4 Review

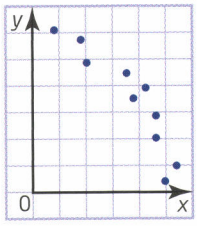
4.1 Investigate Non-Linear Relations, pages 164–167

1. Which scatter plot(s) could be modelled using a curve instead of a line of best fit? Explain.

a)



b)



2. A scientist tested the strength of wood beams by securing beams of various lengths and placing a 500-kg mass at the end of each beam. The table shows the mean deflections, in centimetres.

Length (m)	Deflection (cm)
1.0	0.33
1.5	1.48
2.0	2.51
2.5	4.22
3.0	6.11
3.5	8.17
4.0	10.52
4.5	12.98
5.0	16.72

a) Make a scatter plot of the data. Draw a curve of best fit.

b) Describe the relationship between the length of the beam and the deflection.

c) Use your curve of best fit to predict the deflection of a 6.0-m-long beam.

4.2 Quadratic Relations, pages 168–173

3. Use finite differences to determine whether each relation is linear, quadratic, or neither.

a)

x	y
1	11
2	18
3	27
4	38
5	51

b)

x	y
−2	−10
−1	−2
0	0
1	2
2	10

c)

x	y
−2	−9
−1	−6
0	−3
1	0
2	3

4. The flight of an aircraft from Toronto to Montréal can be modelled by the relation $h = -2.5t^2 + 200t$, where t is the time, in minutes, and h is the height, in metres.

a) Graph the relation.

b) How long does it take to fly from Toronto to Montréal?

c) What is the maximum height of the aircraft? At what time does the aircraft reach this height?

4.3 Investigate Transformations of Quadratics, pages 174–179, and 4.4 Graph $y = a(x - h)^2 + k$, pages 180–188

5. Sketch the graph of each parabola. Describe the transformation from the graph of $y = x^2$.

a) $y = x^2 - 6$

b) $y = -0.5x^2$

c) $y = (x - 2)^2$

d) $y = -2x^2$

6. Copy and complete the table for each parabola. Replace the heading for the second column with the equation for the parabola. Then, sketch each parabola.

Property	$y = a(x - h)^2 + k$
Vertex	
Axis of symmetry	
Stretch or compression factor relative to $y = x^2$	
Direction of opening	
Values x may take	
Values y may take	

a) $y = (x - 1)^2 - 4$

b) $y = 2(x + 3)^2 + 1$

c) $y = \frac{1}{4}(x - 5)^2 + 1$

d) $y = -(x + 2)^2 + 6$

4.5 Quadratic Relations of the Form $y = a(x - r)(x - s)$, pages 189–193

7. Sketch a graph of each quadratic. Label the x-intercepts and the vertex.

a) $y = -(x + 5)(x - 7)$

b) $y = 2(x - 3)(x + 1)$

8. The path of a football can be modelled by the equation $h = -0.0625d(d - 56)$, where h represents the height, in metres, of the football above the ground and d represents the horizontal distance, in metres, of the football from the player.

a) Sketch a graph of this relation.

b) At what horizontal distance does the football land?

c) At what horizontal distance does the football reach its maximum height? What is its maximum height?

4.6 Negative and Zero Exponents, pages 194–201

9. Evaluate.

a) 7^{-2} **b)** 13^{0} **c)** 10^{-5}

d) $(-34)^{0}$ **e)** $(-6)^{-1}$ **f)** $(-7)^{-2}$

g) 6^{0} **h)** $\left(-\frac{2}{5}\right)^{-3}$

10. Joan won a multi-million dollar lottery. She decides to give \$1 000 000 of her winnings to charity. Her plan is to give $\frac{1}{2}$, or 2^{-1}, to charity in January, and then give half of the remaining amount in February, half again in March, and so on.

a) What fraction remains after 6 months?

b) What fraction remains after 12 months?

c) Write each fraction as a power of 2 with a negative exponent.

d) What amount is remaining at the end of the year?

Chapter Problem Wrap-Up

In Section 4.1, question 6, Section 4.2, question 7, and Section 4.4, question 16, you used data from a landfill site to make a scatter plot, draw a curve of best fit, test for a quadratic relation, and determine an equation.

a) Identify the values of x for which your model is accurate.

b) Use your model to predict the total mass of garbage in the landfill in 2020. Comment on the accuracy of your prediction.

c) Analyse how the equation and the graph would change if this city instituted a mandatory recycling program.

d) A statistical analyst suggested that you could use an exponential model of the form $y = 200 \times b^{x}$, where x represents the number of years after 2000. What evidence could justify using this model? Determine an appropriate value of b.

Chapter 4 Practice Test

1. Sketch a graph of each parabola. Label the coordinates of the vertex and the equation of the axis of symmetry.

a) $y = x^2 - 6$

b) $y = 2(x - 5)^2$

c) $y = -\frac{1}{3}(x + 3)^2 + 4$

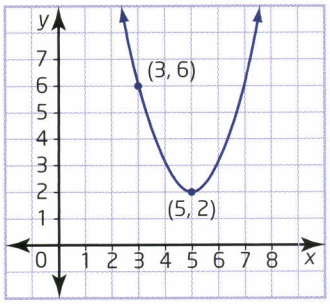
2. Sketch a graph of each relation. Label the x-intercepts and the vertex.

a) $y = (x - 6)(x + 2)$

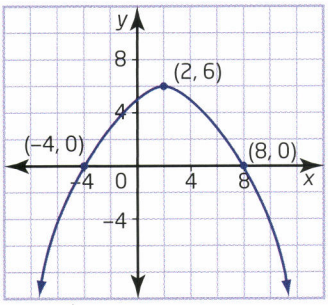
b) $y = -4(x - 1)(x - 9)$

3. Determine an equation to represent each parabola.

a)



b)



4. Evaluate.

a) 4^0

b) 5^{-1}

c) $(-3)^{-3}$

d) $\left(\frac{3}{4}\right)^{-2}$

5. The table shows the length of a spring under a specific load.

Load (kg)	Length (cm)
0	12.0
1	12.6
2	13.8
3	15.6
4	18.0
5	21.0
6	24.6

a) Use finite differences to determine whether this is a quadratic relation.

b) Make a scatter plot of the data. Draw a curve of best fit.

c) Use your curve of best fit to predict the length of the spring under a load of 8 kg.

6. Board-feet are used to measure the total length, in feet, of boards that are 1 inch thick and 1 foot wide that can be cut from a tree to make lumber. You can use the equation $l = 0.011a^2 - 0.68a + 13.31$ to estimate the number of hundreds of board-feet available in a certain type of tree, where l is the number of board-feet and a is the age, in years, of the tree.

a) What is the approximate number of board-feet available to be cut from a 40-year-old tree?

b) What is the approximate number of board-feet available to be cut from a 75-year-old tree?

c) Why does the number of board-feet increase in a quadratic manner as a tree ages?

d) Who would be interested in knowing this type of information? Explain.

7. The St. Louis Gateway Arch in St. Louis, Missouri, was built in 1965 and was designed as a catenary, which is a curve that approximates a parabola. The arch is 192 m wide and 192 m tall.

a) Sketch a graph of the arch that is symmetrical about the y-axis.

b) Label the x-intercepts and the vertex.

c) Determine an equation to model the arch.

8. When a car is travelling at a given speed, there is a minimum turn radius it can safely make. A particular car's minimum radius can be calculated by $r = 0.6s^2$, where s is the speed, in kilometres per hour, and r is the turning radius, in metres. If the car uses tires with better grip, how does this affect the equation? Justify your response.

9. The maximum viewing distance on a clear day is related to how high you are above the surface of Earth. This relationship can be approximated by the formula $h = \frac{3}{40}d^2$, where d is the maximum distance, in kilometres, and h is your height, in metres, above the ground.

a) How high do you need to be in order to see a distance of 25 km?

b) How would the formula change if you were standing on a 20-m cliff?

10. A volleyball's height, h, in metres, above the ground after t seconds is modelled by the relation $h = -4.9t^2 + 5t + 2$.

a) Graph the relation.

b) What is the h-intercept? What does it represent?

c) How long will it take the volleyball to hit the ground? What feature on the graph models this? Explain your answer.

11. An ant colony has 5000 ants on July 1 and doubles every year. This can be expressed as $N = 5000 \times 2^t$, where N represents the number of ants and t represents time, in years.

a) Find the number of ants after 2, 3, 4, and 5 years.

b) What does $t = 0$ represent in this situation? What does $t = -2$ represent?

c) When were there 625 ants? Explain.

12. **Use Technology** The approximate cost of operating a certain car at a constant speed is given by the formula $C = 0.006(s - 50)^2 + 20$, for $10 \leq s \leq 130$, where s is the speed, in kilometres per hour, and C is the cost, in cents per kilometre. Use a graphing calculator to compare the operating costs, at different speeds, to those of a second vehicle with formula $C = 0.008(s - 55)^2 + 15$.

Achievement Check

13. **a)** Graph the following relations by developing a table of values and plotting points. Then, find the first and second differences.

$y = x^2 - 4x$

$y = x^2 - 4x + 5$

$y = x^2 - 4x - 2$

b) Examine each graph and use its properties to write an equation in the form $y = (x - h)^2 + k$.

c) What conclusions can you make about the relation $y = x^2 - 4x + c$ for different values of c?